Implementation of the fair pricing correspondence

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Abstract

This paper proposes an extensive game form that fully implements *the fair pricing correspondence* in the NIMBY problems. The NIMBY problems, namely the not in my back yard problems, consider the problems of deciding a location for a waste disposal facility among districts. For the NIMBY problems, Sakai [2010] has characterized the *fair pricing rules* with a set of interesting axioms. As for the implementation of the rules, Sakai [2010] has pointed out that the rules are not Nash implementable since they violate Maskin monotonicity (Maskin [1999]). This paper considers the fair pricing correspondence, which associates with a NIMBY problem the set of fair pricing rule allocations. Although the fair pricing correspondence is not Nash implementable either, it is implementable in subgame-perfect equilibrium with our two-stage game form.

1. Introduction

This paper proposes a game form that implements the fair pricing correspondence in the NIMBY problems. NIMBY, an acronym of not in my back yard, is often used to describe opposition by residents to locally unwanted public facilities. Examples of the facilities that may cause NIMBY reactions include airports, landfill dump sites, military bases, power plants, prisons, and others. These facilities are necessary public goods for the society. However, for accepters of the facilities, they might be local 'bads' that cause disutilities. Therefore, how to choose a site for a NIMBY facility and how to compensate the accepter of the facility are non-trivial problems.

This paper considers a typical NIMBY problem of deciding a location for a waste disposal facility among districts. For this NIMBY problem, Sakai [2010] has proposed the *fair pricing rules* that possess several desirable properties. The rule chooses an efficient district, whose sum of the disutility and the construction cost of the facility is the smallest among all districts. In the rule, compensation for the accepter of the facility is determined so that each district must make monetary payments in order to share the accepter's disutility and the construction cost in a fair manner, in the sense that each district bears a burden in proportion to the amount of wastes that it produces. Furthermore, the fair pricing rules satisfy axioms such as *core property*, *monotonicity*, and *reallocation-proofness* as Sakai [2010] shows. Sakai also proves that the set of fair pricing rules is characterized by *individual rationality*, *monotonicity*, and *reallocation-proofness* when there are three or more districts in the society. These characterizations of the fair pricing rules indicate the validity and significance of the rules.

When the society, or the social planner, is about to exercise a fair pricing rule, he must collect information on the amount of wastes, the construction cost, and the disutility for each district. Sakai [2010] mentions that the information on the first two items can be collected, but the one on the last item is hard to obtain. So, while the information on the wastes and the costs can be known to the social planner, the information on the disutilities is unknown to him. Pérez-Castrillo and Wettstein [2002] points out that it is often the case that the parties concerned possess much more information than the social planner. For such circumstances, a game form can be used as a tool for the social planner who wishes to implement the rules. The game form itself can be defined independently of the disutilities of the districts in the society. As the literature on implementation theory has proposed, properly designed game forms can realize desirable allocations in equilibrium of the games even if the social planner is given an insufficient amount of information.

As for the implementation of the fair pricing rules, Sakai [2010] has pointed out that the rules are not implementable in Nash equilibrium since they violate Maskin monotonicity (Maskin [1999]), which is a necessary condition for Nash implementation. The present paper considers the fair pricing correspondence, which associates with a NIMBY problem the set of fair pricing rule allocations. Unfortunately, the fair pricing correspondence is not Nash implementable, either. However, the fair pricing correspondence is implementable in subgame-perfect equilibrium with an extensive game form that we propose in the present paper.

Our two-stage game form is relatively simple: In the game form, each district reports just a price, and 'yes' or 'no'. In Stage 1, each district is asked to report a price: The lowest price will be the unit price that each district must pay when it brings one unit of wastes to the facility. In Stage 2, each district is asked whether it wants to accept the facility. If any district says 'yes', the accepter is chosen from those who have reported 'yes'. If all districts say 'no', the accepter is chosen from those who have reported the lowest price in Stage 1. The accepter will bear the construction cost of the facility but it will receive payments from the other districts. Each payment is calculated as a product of the unit price determined in Stage 1 and the amount of wastes that each district produces.

With this game form, every fair pricing rule allocation can be realized as an equilibrium allocation. Moreover, every equilibrium allocation is in fact a fair pricing rule allocation. So, our game form *fully* implements the fair pricing correspondence.

Besides Sakai [2010], several papers are closely related to our research. Ehlers [2009] as well as Pérez-Castrillo and Wettstein [2002] considers a model that can be used for choosing a location of a NIMBY facility. These papers propose multi-bidding game forms and analyze Nash equilibrium allocations. The model of these papers is different from Sakai's in that the latter model puts more structures in the model, such as cost functions and disutility functions that are increasing in the amount of wastes. Minehart and Neeman [2002] consider a model that is closer to Sakai's, but the difference is in that the latter model explicitly distinguishes costs from disutilities. In addition, Minehart and Neeman propose a bidding game form that resembles the second–price auction, and its equilibrium allocations are different from fair pricing rule allocations. In the present paper, we follow the model introduced by Sakai [2010], so our settings are different from the above papers'. Our contribution to the literature is to propose a game form that exactly achieves the fair pricing rule allocations in subgame-perfect equilibria.

The remaining part of this paper is organized as follows. Section 2 defines the NIMBY problems and introduces the notions of the fair pricing correspondence and implementation. Section 3 proposes a game form and proves the result. Section 4 provides some concluding remarks. The appendix proves that the fair pricing correspondence is not implementable in Nash equilibrium.

2. The model

The model follows Sakai [2010]. The set of *districts* is denoted by $N \equiv \{1, 2, \ldots, n\}$ where $n \geq 2$. Each district $i \in N$ produces an amount $w_i \geq 0$ of wastes. Let $w \equiv (w_i)_{i \in N}$ be a profile of waste parameters. The total amount of wastes is denoted by $W \equiv \sum_{i \in N} w_i$. We assume that $W > 0$. Let $\mathcal{W} \equiv \{w \in \mathbb{R}_+^N : W > 0\}$ be the set of profiles of waste parameters.

The wastes should be disposed at a facility, which is to be constructed at some district. The construction cost of the facility with a capacity W at district i is given by $c_i(W) \geq 0$. The *cost function* $c_i\colon \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to be weakly concave, strictly increasing, and satisfying $c_i(0) = 0$. The set of cost functions is denoted by $\mathcal{C}$. Let $c \equiv (c_i)_{i \in N}$ be a profile of cost functions. The set of profiles of cost functions is denoted by $\mathcal{C}^N$

When an amount W of wastes is disposed at district i's facility, it bears a disutility $v_i(W) \geq 0$. The *disutility function* $v_i\colon \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to be weakly concave, strictly increasing, and satisfying $v_i(0) = 0$. The set of disutility functions is denoted by $\mathcal{V}$. The significance of distinguishing c_i from v_i is discussed in Sakai [2010]. Let $v \equiv (v_i)_{i \in N}$ be a profile of disutility functions. The set of profiles of disutility functions is denoted by $\mathcal{V}^N$

If district i's facility deals with an amount W of wastes and i receives a net monetary transfer $m_i \in \mathbb{R}$, then i obtains a utility

$$u_i(W, m_i) \equiv -v_i(W) + m_i.$$

Let $m \equiv (m_i)_{i \in N}$ be a profile of net monetary transfers.

A NIMBY problem is a list

$$(w, v, c) \in \mathcal{D} \equiv \mathcal{W} \times \mathcal{V}^N \times \mathcal{C}^N.$$

An *assignment function*, $\sigma\colon N \to \{0, 1\}$ satisfying $|\sigma^{-1}(1)| = 1$, specifies whether or not a facility is assigned to a given district. If $\sigma(j) = 1$, then it means that a facility is to be constructed at district j and all the other districts will not have any facilities: District j is called the *accepter*. The other districts are called *non-accepters*. The set of assignment functions is denoted by $\mathcal{A}$.

An *allocation* x for $(w, c) \in \mathcal{W} \times \mathcal{C}^N$ is a list

$$x \equiv (W, \sigma, m) \in \{W\} \times \mathcal{A} \times \mathbb{R}^N$$

satisfying the budget balance condition $c_j(W) = -\sum_{i \in N} m_i$ where $j = \sigma^{-1}(1)$. The budget balance condition says that when a facility is constructed at district j, the exact amount of the construction cost, $c_j(W)$, is covered by the sum of net monetary payments from each district,

$-\sum_{i \in N} m_i$. For an allocation x, let x_i denote district i's bundle

$$x_i \equiv (\sigma(i) \cdot W, m_i) \in \mathbb{R}_+ \times \mathbb{R}.$$

For example, if district j is the accepter, then $x_j = (W, m_j)$ and $x_i = (0, m_i)$ for each $i \neq j$. The set of allocations for $(w, c) \in \mathcal{W} \times \mathcal{C}^N$ is denoted by $X(w, c)$.

The *fair price* for $(w, v, c) \in \mathcal{D}$ is defined by

$$p(w, v, c) \equiv \frac{\min_{i \in N}(v_i(W) + c_i(W))}{W}.$$

A *fair pricing rule* ψ is a single-valued function that associates with each $(w, v, c) \in \mathcal{D}$ an allocation $\psi(w, v, c) = (W, \sigma, m) \in X(w, c)$ such that

$$j \in \arg\min_{i \in N}(v_i(W) + c_i(W)),$$

$$\sigma(j) = 1 \text{ and } \sigma(i) = 0 \text{ for each } i \neq j,$$

$$m_i = -p(w, v, c) \cdot w_i + \sigma(i) \cdot v_i(W) \text{ for each } i \in N.$$

Let $\psi_i(w, v, c) = (\sigma(i) \cdot W, m_i)$ denote district i's bundle for $\psi(w, v, c)$. The rule says that the accepter should be district j who is *efficient* in the sense that district j minimizes the sum of the disutility and the construction cost for the amount W of wastes.[1] The rule also says that each district must make monetary payments in order to share j's disutility and the construction cost in a fair manner in the following sense: For $\psi(w, v, c)$, each district $i \in N$ obtains a utility

$$u_i(\psi_i(w, v, c)) = -\frac{w_i}{W}(v_j(W) + c_j(W))$$

so that each district bears a burden in proportion to the amount of wastes that it produces.

We remark that there exist multiple fair pricing rules because if there are multiple efficient districts (i.e., $|\arg\min_{i \in N}(v_i(W) + c_i(W))| > 1$), then there are multiple assignment functions that choose a single efficient district. Let Ψ be the set of fair pricing rules. We say that an allocation $x \in X(w, c)$ is a *fair pricing rule allocation* for $(w, v, c) \in D$ if $x = \psi(w, v, c)$ for some $\psi \in \Psi$.

[1] Note that creating multiple facilities with capacities less than W cannot be more efficient than creating one facility with a capacity W at district $j \in \arg\min_{i \in N}(v_i(W) + c_i(W))$. This is because, in our model, $v_i(W) + c_i(W)$ is weakly concave, strictly increasing, and satisfying $v_i(0) + c_i(0) = 0$ for each $i \in N$.

The *fair pricing correspondence* φ is a multi-valued function that associates with each $(w, v, c) \in \mathcal{D}$ the set of fair pricing rule allocations,

$$\varphi(w, v, c) \equiv \{x \in X(w, c) : x = \psi(w, v, c) \text{ for some } \psi \in \Psi\}.$$

Note that the fair pricing correspondence associates the same allocation as a fair pricing rule when the efficient district is unique.

Sakai [2010] shows that any fair pricing rule satisfies the following list of axioms; the *core property*, *monotonicity*, and *reallocation-proofness*. Sakai [2010] also shows that the set of fair pricing rules is characterized by *individual rationality*, *monotonicity*, and *reallocation-proofness* when $n \geq 3$. These characterizations of the fair pricing rules indicate the validity and significance of the rules.

When the social planner is about to exercise a fair pricing rule, he must collect information on the amount of wastes, the construction cost, and the disutility for each district. Sakai [2010] mentions that the information on the first two items can be collected, but the one on the last item is hard to obtain. Pérez-Castrillo and Wettstein [2002] points out that it is often the case that the parties concerned possess much more information than the social planner. For such circumstances, a game form can be used as a tool for the uninformed social planner who wishes to implement the rules. As the literature on implementation theory has proposed, properly designed game forms can realize desirable allocations in equilibrium of the games even if the social planner is given an insufficient amount of information.[2]

As for the implementation of the fair pricing rules, Sakai [2010] has pointed out that the rules are not Nash implementable since they violate Maskin monotonicity (Maskin [1999]), which is a necessary condition for Nash implementation. The present paper considers the fair pricing correspondence, which is not Nash implementable either, as is discussed in the appendix. However, it is implementable in subgame-perfect equilibrium as we propose in the present paper.[3]

We consider a two-stage extensive game form $\Gamma(w, c)$ with perfect information. Following Sakai [2010], we assume that w and c are known but v is unknown to the social planner when we consider the implementation problem.

The *game form* $\Gamma(w, c)$ consists of a game tree with the set of choices available to districts at

[2]For a survey on implementation theory, readers are referred to Jackson [2001].

[3]Necessary and sufficient conditions for subgame-perfect implementation are studied in Moore and Repullo [1988].

each decision node, and an outcome function O. A list s_i of district i's choice at each decision node is called district i's *strategy*. For a strategy profile $s \equiv (s_1, s_2, \dots, s_n)$, the corresponding outcome allocation is denoted by $O(s) \in X(w, c)$.

Given $v \in \mathcal{V}^N$, a pair $(\Gamma(w, c), v)$ constitutes an extensive form *game*. A strategy profile s is called a *subgame-perfect equilibrium* of $(\Gamma(w, c), v)$ if the choices specified in the strategy profile constitute a Nash equilibrium in every subgame of $(\Gamma(w, c), v)$. The set of outcome allocations corresponding to pure-strategy subgame-perfect equilibria of $(\Gamma(w, c), v)$ is denoted by $SPE(\Gamma(w, c), v)$.

We say that a game form $\Gamma(w, c)$ *fully implements the fair pricing correspondence* φ *in subgame-perfect equilibrium* if $\varphi(w, v, c) = SPE(\Gamma(w, c), v)$ for all $v \in \mathcal{V}^N$. The full implementation requires that every fair pricing rule allocation can be realized as an equilibrium allocation as well as every equilibrium allocation is in fact a fair pricing rule allocation.

3. Result

3.1. The game form

This section presents a two-stage game form that implements the fair pricing correspondence in subgame-perfect equilibrium. Given $(w, c) \in \mathcal{W} \times \mathcal{C}^N$, which is assumed to be known to the social planner, the game form $\Gamma(w, c)$ is defined as follows.

Stage 1. Each district $i \in N$ simultaneously reports $p_i > 0$. Let $p = (p_1, p_2, \dots, p_n)$.

Stage 2. After observing p, each district $i \in N$ simultaneously reports $q_i(p) \in \{\text{'yes'}, \text{'no'}\}$. Each district i's strategy is denoted by $s_i = (p_i, q_i(\cdot))$.

The outcome function. Let $p^* = \min_{i \in N} p_i$. The outcome allocation $O(s) = (W, \sigma, m)$ is such that $\sigma(j) = 1$, $\sigma(i) = 0$ and $m_i = -p^* \cdot w_i$ for each $i \neq j$, and $m_j = -c_j(W) - \sum_{i \neq j} m_i$, where the accepter j is chosen by the following criteria.

Criterion 1: If any district reports 'yes' in Stage 2, then let j be the district with the largest index among those who have reported the highest price in Stage 1 among those who have reported 'yes' in Stage 2: $j = \max(\arg\max_{k \in \{i \in N : q_i(p) = \text{'yes'}\}} p_k)$.

Criterion 2: If all districts report 'no' in Stage 2, then let j be the district with the least index among those who have reported p^* in Stage 1: $j = \min(\arg\min_{i \in N} p_i)$.

Note that the game form is defined independently of $v \in \mathcal{V}^N$. In words, the game form is described as follows. In Stage 1, each district is asked to report a price: The lowest price p^* will be the unit price that each district must pay when it brings one unit of wastes to a facility. In Stage 2, each district is asked whether it wants to accept the facility. If any district says 'yes' (Criterion 1), the accepter is chosen from those who have reported the highest price among those who have reported 'yes' in Stage 2. If all districts say 'no' (Criterion 2), the accepter is chosen from those who have reported the lowest price p^* in Stage 1. The accepter j will bear the construction cost $c_j(W)$ of the facility and receive payments, which sum up to $p^* \cdot (W - w_j)$, from the other districts.

The result of the present paper is the following.

Theorem. *The game form $\Gamma(w, c)$ fully implements the fair pricing correspondence φ in subgame-perfect equilibrium. That is, for all $v \in \mathcal{V}^N$, we have $\varphi(w, v, c) = SPE(\Gamma(w, c), v)$.*

3.2. Proof

3.2.1. $\varphi(w, v, c) \subset SPE(\Gamma(w, c), v)$

Fix $(w, c) \in \mathcal{W} \times \mathcal{C}^N$ and take any $v \in \mathcal{V}^N$. Take any fair pricing rule allocation $x = (W, \sigma, m) \in \varphi(w, v, c)$. We have $x = \psi(w, v, c)$ for some $\psi \in \Psi$.

Let $j = \sigma^{-1}(1)$ be the accepter for the allocation x. Define the strategy profile $s = (s_1, s_2, \ldots, s_n)$ as follows.

Each $i \in N$ chooses $\bar{p}_i = p(w, v, c)$ in Stage 1. Let $\bar{p} = (\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n)$.

Each district's choice in Stage 2 after observing $p = (p_1, p_2, \ldots, p_n)$ reported in Stage 1 is described as follows. Let $p^* = \min_{i \in N} p_i$.

For $j = \sigma^{-1}(1)$, $q_j(\cdot)$ is such that

$$q_j(p) = \text{'yes' if } p^* \geq \frac{v_j(W) + c_j(W)}{W}, \text{ and } q_j(p) = \text{'no' otherwise.}$$

For each $i \neq j$, $q_i(\cdot)$ is such that

$$q_i(p) = \text{'yes' if } p^* > \frac{v_i(W) + c_i(W)}{W}, \text{ and } q_i(p) = \text{'no' otherwise.}$$

Lemma 1. *For the strategy profile s, the outcome allocation $O(s) = (W', \sigma', m')$ is equal to $x = (W, \sigma, m)$.*

Proof. First, $W' = W$ is obvious by the definition of allocations.

Second, note that

$$p^* = \min_{i \in N} \bar{p}_i = p(w, v, c) = \frac{\min_{i \in N}(v_i(W) + c_i(W))}{W}$$

according to s. Since j is the accepter for the fair pricing rule allocation x, we have $j \in \arg\min_{i \in N}(v_i(W) + c_i(W))$. Hence $q_j(\bar{p}) =$ 'yes' according to s_j. For each $i \neq j$, the condition

$$p^* > \frac{v_i(W) + c_i(W)}{W}$$

never holds, so $q_i(\bar{p}) =$ 'no' according to s_i. Therefore, Criterion 1 of the game form $\Gamma(w, c)$ applies and we have $\sigma' = \sigma$.

Third, for each $i \neq j$, we have $m'_i = -p^* \cdot w_i = -p(w, v, c) \cdot w_i = m_i$ since $\sigma(i) = 0$. For the accepter j,

$$\begin{aligned}
m'_j &= -c_j(W) - \sum_{i \neq j} m'_i \\
&= -c_j(W) + p(w, v, c) \cdot (W - w_j) \\
&= -c_j(W) + \frac{v_j(W) + c_j(W)}{W} \cdot W - p(w, v, c) \cdot w_j \\
&= -p(w, v, c) \cdot w_j + v_j(W) \\
&= m_j
\end{aligned}$$

since $\sigma(j) = 1$. Therefore, $m' = m$ and we have $O(s) = (W', \sigma', m') = (W, \sigma, m) = x$. □

We now show that the strategy profile s is a subgame-perfect equilibrium of the game $(\Gamma(w, c), v)$.

Lemma 2. *Consider the subgame that starts at Stage 2 after $p = (p_1, p_2, \ldots, p_n)$ has been reported in Stage 1. Suppose that each $i \in N$ in this subgame takes strategy $q_i(p)$ induced by s_i. The strategy profile $q(p) = (q_1(p), q_2(p), \ldots, q_n(p))$ is a Nash equilibrium of this subgame.*

Proof. Let $p^* = \min_{i \in N} p_i$. Let k be the accepter for the outcome allocation for the strategy profile $q(p)$.

First, we note that k cannot gain by any deviation from $q_k(p)$ to $q'_k(p)$. Let y be the outcome allocation before the deviation and y' be the outcome allocation after the deviation. If $q_k(p) =$ 'no' then the accepter remains k even after the deviation, and hence $y = y'$: So k cannot gain by the deviation. If $q_k(p) =$ 'yes' then the accepter after the deviation is either k or another district $i \neq k$. Since k cannot gain as long as he remains the accepter, let us consider the case where the accepter changes into $i \neq k$ after the deviation. Before the deviation, k obtains utility

$$u_k(y_k) = -v_k(W) - c_k(W) + p^* \cdot (W - w_k)$$

as the accepter. After the deviation, k obtains utility

$$u_k(y'_k) = -p^* \cdot w_k$$

as a non-accepter. Since $q_k(p) =$ 'yes' is induced by s_k, we have

$$\begin{aligned} & p^* \geq \frac{v_k(W) + c_k(W)}{W}, \\ \Longrightarrow \quad & -v_k(W) - c_k(W) + p^* \cdot (W - w_k) \geq -p^* \cdot w_k, \\ \Longrightarrow \quad & u_k(y_k) \geq u_k(y'_k), \end{aligned}$$

so k cannot gain by the deviation anyway.

Second, we note that each non-accepter $i \neq k$ cannot gain by any deviation from $q_i(p)$ to $q'_i(p)$. Let y be the outcome allocation before the deviation and y' be the outcome allocation after the deviation. If $q_i(p) =$ 'yes' then i remains a non-accepter even after the deviation, and hence $y = y'$: So i cannot gain by the deviation. If $q_i(p) =$ 'no' then i may or may not be the accepter after the deviation. Since i cannot gain as long as he remains a non-accepter, let us consider the case where i becomes the accepter after the deviation. Before the deviation, i obtains utility

$$u_i(y_i) = -p^* \cdot w_i$$

as a non-accepter. After the deviation, i obtains utility

$$u_i(y'_i) = -v_i(W) - c_i(W) + p^* \cdot (W - w_i)$$

as the accepter. Since $q_i(p) =$ 'no' is induced by s_i, we have

$$\begin{aligned} & p^* \leq \frac{v_i(W) + c_i(W)}{W}, \\ \Longrightarrow \quad & -v_i(W) - c_i(W) + p^* \cdot (W - w_i) \leq -p^* \cdot w_i, \\ \Longrightarrow \quad & u_i(y_i') \leq u_i(y_i), \end{aligned}$$

so i cannot gain by the deviation anyway.

Since all districts cannot gain by any unilateral deviation, the strategy profile $q(p)$ is a Nash equilibrium of the subgame of our concern. □

Lemma 3. *The strategy profile s is a subgame-perfect equilibrium of the game $(\Gamma(w,c), v)$.*

Proof. We show that the strategy profile s is a Nash equilibrium of the game $(\Gamma(w,c), v)$, which, together with Lemma 2, proves the present lemma. It is sufficient to show that for each $i \in N$, his strategy $s_i = (\bar{p}_i, q_i(\cdot))$ is a best response to the strategies of the others, $(s_k)_{k \neq i}$, where $s_k = (\bar{p}_k, q_k(\cdot))$.

By contradiction, suppose that district i's deviation from s_i to $s_i' = (p_i', q_i'(\cdot))$ is profitable for him. Let $(p_i', \bar{p}_{-i})$ be a vector that is obtained by replacing the i-th component of $\bar{p} = (\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n)$ with p_i'. Since Lemma 2 implies that $q_i(p_i', \bar{p}_{-i})$ is a best response to $(q_k(p_i', \bar{p}_{-i}))_{k \neq i}$ in the subgame starting at Stage 2 with $(p_i', \bar{p}_{-i})$, district i's another deviation from s_i to $s_i'' = (p_i', q_i(\cdot))$ is also profitable for him. Henceforth, we focus on the latter deviation from s_i to s_i''. Note that we must have $p_i' \neq \bar{p}_i$.

First, consider the case where $p_i' > \bar{p}_i$. In this case,

$$\min\{p_i', \min_{k \in N \setminus \{i\}} \bar{p}_k\} = \min_{k \in N} \bar{p}_k = p(w, v, c) = \frac{\min_{k \in N}(v_k(W) + c_k(W))}{W}.$$

Therefore, $q_j(p_i', \bar{p}_{-i}) = q_j(\bar{p}) =$ 'yes' for $j = \sigma^{-1}(1)$, and $q_k(p_i', \bar{p}_{-i}) = q_k(\bar{p}) =$ 'no' for each $k \neq j$. Hence the outcome allocation remains unchanged after the deviation. This contradicts the fact that s_i'' is a profitable deviation.

Next, consider the case where $p_i' < \bar{p}_i$. Let x' be the outcome allocation after the deviation, and recall that, by Lemma 1, x is the outcome allocation before the deviation. Note that i becomes the accepter after the deviation because

$$\min\{p_i', \min_{k \in N \setminus \{i\}} \bar{p}_k\} = p_i' < \min_{k \in N} \bar{p}_k = p(w, v, c) = \frac{\min_{k \in N}(v_k(W) + c_k(W))}{W}$$

and hence $q_k(p'_i, \bar{p}_{-i}) =$ 'no' for each $k \in N$. Criterion 2 of the game form applies after the deviation, and i becomes the accepter.

If i is the accepter both before and after the deviation, then

$$
\begin{aligned}
& p'_i < p(w,v,c), \\
\Longrightarrow \quad & -v_i(W) - c_i(W) + p'_i \cdot (W - w_i) \leq -v_i(W) - c_i(W) + p(w,v,c) \cdot (W - w_i), \\
\Longrightarrow \quad & u_i(x'_i) \leq u_i(x_i),
\end{aligned}
$$

which contradicts the fact that s''_i is a profitable deviation.

If i is a non-accepter before the deviation and becomes the accepter after the deviation, then

$$
\begin{aligned}
& p'_i < p(w,v,c) = \frac{\min_{k \in N}(v_k(W) + c_k(W))}{W} \leq \frac{v_i(W) + c_i(W)}{W}, \\
\Longrightarrow \quad & -v_i(W) - c_i(W) + p'_i \cdot (W - w_i) \leq -v_i(W) - c_i(W) + p(w,v,c) \cdot (W - w_i) \\
& \leq -p(w,v,c) \cdot w_i, \\
\Longrightarrow \quad & u_i(x'_i) \leq u_i(x_i),
\end{aligned}
$$

which contradicts the fact that s''_i is a profitable deviation.

We have proved that for each $i \in N$, s_i is a best response to $(s_k)_{k \neq i}$. Therefore, the strategy profile s is a Nash equilibrium, which, together with Lemma 2, implies that s is a subgame-perfect equilibrium. $\square$

So far, we have shown the following: For any fair pricing rule allocation $x \in \varphi(w,v,c)$, we can construct the strategy profile s such that $O(s) = x$ (Lemma 1) and s is a subgame-perfect equilibrium of the game $(\Gamma(w,c), v)$ (Lemma 3). Therefore, the following proposition holds.

Proposition 1. *For all $v \in \mathcal{V}^N$, we have $\varphi(w,v,c) \subset SPE(\Gamma(w,c),v)$.*

3.2.2. $SPE(\Gamma(w,c),v) \subset \varphi(w,v,c)$

Fix $(w,c) \in \mathcal{W} \times \mathcal{C}^N$ and take any $v \in \mathcal{V}^N$. Given the fair price $p(w,v,c)$, whenever $p(w,v,c) < (v_i(W) + c_i(W))/W$ for some $i \in N$, we define

$$
p^{**} = \min\{z \in \mathbb{R} : z > p(w,v,c) \text{ and } z = \frac{v_i(W) + c_i(W)}{W} \text{ for some } i \in N\}.
$$

Note that p^{**} is undefined when all districts are efficient.

Take any subgame-perfect equilibrium allocation $x = (W, \sigma, m) \in SPE(\Gamma(w, c), v)$. We have $x = O(s)$ for some strategy profile s that is a subgame-perfect equilibrium of the game $(\Gamma(w, c), v)$. Let $j = \sigma^{-1}(1)$ be the accepter for the allocation x. For each $i \in N$, let p_i and $q_i(\cdot)$ be district i's choices induced by s_i. Let $p^* = \min_{i \in N} p_i$.

Lemma 4. *Suppose that $p(w, v, c) < (v_i(W) + c_i(W))/W$ for some $i \in N$ and p^{**} is well-defined. Consider the subgame that starts at Stage 2 after $\hat{p} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n)$ has been reported in Stage 1. Suppose that for $\hat{p}^* = \min_{i \in N} \hat{p}_i$, we have $p(w, v, c) < \hat{p}^* < p^{**}$. Let k be the accepter for the outcome allocation y in this subgame for the strategy profile $q(\hat{p}) = (q_1(\hat{p}), q_2(\hat{p}), \ldots, q_n(\hat{p}))$ induced by s. Then $k \in \arg\min_{i \in N}(v_i(W)+c_i(W))/W$. That is, district k is efficient and $(v_k(W) + c_k(W))/W = p(w, v, c)$.*

Proof. Since $q(\hat{p})$ is induced by s that is a subgame-perfect equilibrium of the original game, $q(\hat{p})$ is a Nash equilibrium of the subgame of our concern.

By contradiction, suppose that $k \notin \arg\min_{i \in N}(v_i(W) + c_i(W))/W$. Then, it must be the case that

$$\begin{aligned} & \frac{v_k(W) + c_k(W)}{W} \geq p^{**} > \hat{p}^*, \\ \Longrightarrow \quad & -\hat{p}^* \cdot w_k > -v_k(W) - c_k(W) + \hat{p}^* \cdot (W - w_k), \\ \Longrightarrow \quad & -\hat{p}^* \cdot w_k > u_k(y_k). \end{aligned}$$

That is, for the price $\hat{p}^*$ used in Stage 2, district k obtains higher utility as a non-accepter than as the accepter. This implies that district k cannot become a non-accepter by any deviation from $q_k(\hat{p})$ since $q(\hat{p})$ is a Nash equilibrium of the subgame. So, we obtain the following two conditions: (1) $q_i(\hat{p}) =$ 'no' for each $i \neq k$,[4] and (2) $k = \min(\arg\min_{i \in N} \hat{p}_i)$.[5]

Now, consider $\ell \in \arg\min_{i \in N}(v_i(W) + c_i(W))/W$. Clearly, $\ell \neq k$. Note that

$$\begin{aligned} & \frac{v_\ell(W) + c_\ell(W)}{W} = p(w, v, c) < \hat{p}^*, \\ \Longrightarrow \quad & -\hat{p}^* \cdot w_\ell < -v_\ell(W) - c_\ell(W) + \hat{p}^* \cdot (W - w_\ell), \\ \Longrightarrow \quad & u_\ell(y_\ell) < -v_\ell(W) - c_\ell(W) + \hat{p}^* \cdot (W - w_\ell). \end{aligned}$$

[4] If $q_i(\hat{p}) =$ 'yes' for some $i \neq k$, district k can become a non-accepter by a deviation to $q'_k(\hat{p}) =$ 'no' by Criterion 1.

[5] Under the condition (1), if $k \neq \min(\arg\min_{i \in N} \hat{p}_i)$, then district k can become a non-accepter by a deviation to $q'_k(\hat{p}) =$ 'no' by Criterion 2.

That is, for the price $\hat{p}^*$ used in Stage 2, district ℓ obtains higher utility as the accepter than as a non-accepter. This implies that district ℓ cannot become the accepter by any deviation from $q_\ell(\hat{p})$ since $q(\hat{p})$ is a Nash equilibrium of the subgame. So, we obtain the following three conditions: (3) $q_k(\hat{p}) =$ 'yes',[6] (4) $\hat{p}_k = \hat{p}_\ell$,[7] and (5) $k > \ell$.[8] However, the conditions (2), (4), and (5) cannot hold simultaneously because the conditions (2) and (4) imply that $k \leq \ell$. A contradiction obtains. $\square$

We now characterize a subgame-perfect equilibrium allocation $x = (W, \sigma, m)$. Recall that $p^* = \min_{i \in N} p_i$ is the price used in Stage 2 for a subgame-perfect equilibrium s associated with the allocation x. Note that $j = \sigma^{-1}(1)$ is the accepter for the allocation x.

Lemma 5. *If $W > w_j$, then $p^* \leq p(w, v, c)$.*

Proof. By contradiction, suppose that $p^* > p(w, v, c)$. Take a non-accepter $i \neq j$ such that $w_i > 0$. Since $W > w_j$, there exists such a non-accepter i.

For the case where $p(w, v, c) < (v_i(W) + c_i(W))/W$, consider i's deviation from $s_i = (p_i, q_i(\cdot))$ to $s'_i = (p'_i, q_i(\cdot))$ such that $p(w, v, c) < p'_i < \min\{p^*, p^{**}\}$. Let $x' = (W, \sigma', m')$ be the outcome allocation after the deviation. By Lemma 4, $\sigma'(i) = 0$, i.e., i remains a non-accepter after the deviation. In this case, i can gain by the deviation because

$$\begin{aligned} & p'_i < p^*, \\ \Longrightarrow \quad & -p'_i \cdot w_i > -p^* \cdot w_i, \\ \Longrightarrow \quad & u_i(x'_i) > u_i(x_i). \end{aligned}$$

This contradicts the fact that s is a subgame-perfect equilibrium.

For the case where $p(w, v, c) = (v_i(W) + c_i(W))/W$, consider i's deviation from $s_i = (p_i, q_i(\cdot))$ to $s'_i = (p'_i, q_i(\cdot))$ such that $p(w, v, c) < p'_i < p^*$. Let $x' = (W, \sigma', m')$ be the outcome allocation after the deviation. If i remains a non-accepter after the deviation, we obtain a contradiction by a similar argument to the previous paragraph. If i becomes the accepter after

[6] Under the condition (1), if $q_k(\hat{p}) =$ 'no', then district ℓ can become the accepter by a deviation to $q'_\ell(\hat{p}) =$ 'yes' by Criterion 1.

[7] By the condition (2), $\hat{p}_k \leq \hat{p}_\ell$. Under the conditions (1) and (3), if $\hat{p}_k < \hat{p}_\ell$, then district ℓ can become the accepter by a deviation to $q'_\ell(\hat{p}) =$ 'yes' by Criterion 1.

[8] If $k \leq \ell$, then $k < \ell$ since $k \neq \ell$. Under the conditions (1), (3), and (4) and $k < \ell$, district ℓ can become the accepter by a deviation to $q'_\ell(\hat{p}) =$ 'yes' by Criterion 1.

the deviation, i can gain by the deviation because

$$\begin{aligned} & p(w,v,c) = \frac{v_i(W)+c_i(W)}{W} < p'_i < p^*, \\ \Longrightarrow\ & -v_i(W) - c_i(W) + p'_i \cdot (W - w_i) > -p'_i \cdot w_i > -p^* \cdot w_i, \\ \Longrightarrow\ & u_i(x'_i) > u_i(x_i). \end{aligned}$$

This contradicts the fact that s is a subgame-perfect equilibrium. □

Lemma 6. *If* $W > w_j$, *then* $p^* \geq (v_j(W) + c_j(W))/W$.

Proof. By contradiction, suppose that $p^* < (v_j(W) + c_j(W))/W$. Consider j's deviation from $s_j = (p_j, q_j(\cdot))$ to $s'_j = (p'_j, q'_j(\cdot))$ such that $p'_j = (v_j(W) + c_j(W))/W$ and $q'_j(p'_j, p_{-j}) =$ 'no', where (p'_j, p_{-j}) denotes a vector that is obtained by replacing the j-th component of $(p_1, p_2, \ldots, p_n)$ with p'_j. Let x' be the outcome allocation after the deviation.

First, suppose that j remains the accepter after the deviation. Since $q'_j(p'_j, p_{-j}) =$ 'no', it must be the case that Criterion 2 of the game form applies and $p'_j = \min\{p'_j, \min_{i \in N \setminus \{j\}} p_i\}$. Note that the price used in Stage 2 after the deviation is p'_j. In this case, j can gain by the deviation because

$$\begin{aligned} & p^* < p'_j, \\ \Longrightarrow\ & -v_j(W) - c_j(W) + p^* \cdot (W - w_j) < -v_j(W) - c_j(W) + p'_j \cdot (W - w_j), \\ \Longrightarrow\ & u_j(x_j) < u_i(x'_j). \end{aligned}$$

This contradicts the fact that s is a subgame-perfect equilibrium.

Second, suppose that j becomes a non-accepter after the deviation. Let p'' be the price used in Stage 2 after the deviation, i.e., $p'' = \min\{p'_j, \min_{i \in N \setminus \{j\}} p_i\}$. Note that either $[\, p^* < p'' = p'_j \,]$ or $[\, p^* \leq p'' < p'_j \,]$ holds. In both cases, j can gain by the deviation because

$$\begin{aligned} & p^* < p'' = p'_j = \frac{v_j(W)+c_j(W)}{W}, \\ \Longrightarrow\ & -v_j(W) - c_j(W) + p^* \cdot (W - w_j) < -v_j(W) - c_j(W) + p'' \cdot (W - w_j) = -p'' \cdot w_j, \\ \Longrightarrow\ & u_j(x_j) < u_j(x'_j), \end{aligned}$$

and

$$
\begin{aligned}
& p^* \leq p'' < p'_j = \frac{v_j(W) + c_j(W)}{W}, \\
\Longrightarrow \quad & -v_j(W) - c_j(W) + p^* \cdot (W - w_j) \leq -v_j(W) - c_j(W) + p'' \cdot (W - w_j) < -p'' \cdot w_j, \\
\Longrightarrow \quad & u_j(x_j) < u_j(x'_j).
\end{aligned}
$$

This contradicts the fact that s is a subgame-perfect equilibrium. □

Lemma 7. *If $W > w_j$, then $x \in \varphi(w, v, c)$.*

Proof. By Lemmas 5 and 6, we have

$$
\frac{v_j(W) + c_j(W)}{W} \leq p^* \leq p(w, v, c) = \frac{\min_{i \in N}(v_i(W) + c_i(W))}{W}
$$

and hence $p^* = p(w, v, c)$ and $j \in \arg\min_{i \in N}(v_i(W) + c_i(W))$. The choice of j at the beginning of the present section ensures that $\sigma(j) = 1$ and $\sigma(i) = 0$ for each $i \neq j$. The outcome function of the game form ensures that $m_i = -p(w, v, c) \cdot w_i$ for each $i \neq j$ and

$$
\begin{aligned}
m_j &= -c_j(W) - \sum_{i \neq j} m_i \\
&= -c_j(W) + p(w, v, c) \cdot (W - w_j) \\
&= -c_j(W) + \frac{v_j(W) + c_j(W)}{W} \cdot W - p(w, v, c) \cdot w_j \\
&= -p(w, v, c) \cdot w_j + v_j(W)
\end{aligned}
$$

Therefore, x is a fair pricing rule allocation and hence $x \in \varphi(w, v, c)$. □

Lemma 8. *If $W = w_j$, then $j \in \arg\min_{i \in N}(v_i(W) + c_i(W))$.*

Proof. By contradiction, suppose that $j \notin \arg\min_{i \in N}(v_i(W) + c_i(W))$, that is, district j is not efficient, $p(w, v, c) < (v_j(W) + c_j(W))/W$, and p^{**} is well-defined. Define $p^*_{-j} = \min_{i \in N \setminus \{j\}} p_i$.

For the case where $p^*_{-j} \leq p(w, v, c)$, consider j's deviation from $s_j = (p_j, q_j(\cdot))$ to $s'_j = (p'_j, q'_j(\cdot))$ such that $p'_j > p^*_{-j}$ and $q'_j(p'_j, p_{-j}) =$ 'no'. Let x' be the outcome allocation after the deviation. Note that j becomes a non-accepter after the deviation no matter which criterion of the game form, Criterion 1 or 2, may apply. Furthermore, the price used in Stage 2 after the deviation is p^*_{-j}. In this case, j can gain by the deviation because

$$\begin{aligned} & p^*_{-j} \leq p(w,v,c) < \frac{v_j(W)+c_j(W)}{W}, \\ \Longrightarrow \quad & -p^*_{-j} \cdot W > -v_j(W) - c_j(W), \\ \Longrightarrow \quad & u_j(x'_j) > u_j(x_j). \end{aligned}$$

This contradicts the fact that s is a subgame-perfect equilibrium.

For the case where $p^*_{-j} > p(w,v,c)$, consider j's deviation from $s_j = (p_j, q_j(\cdot))$ to $s'_j = (p'_j, q_j(\cdot))$ such that $p(w,v,c) < p'_j < \min\{p^{**}, p^*_{-j}\}$. Let x' be the outcome allocation after the deviation. By Lemma 4, district j, that is not efficient, becomes a non-accepter after the deviation. Note that the price used in Stage 2 after the deviation is p'_j. In this case, j can gain by the deviation because

$$\begin{aligned} & p'_j < p^{**} \leq \frac{v_j(W)+c_j(W)}{W}, \\ \Longrightarrow \quad & -p'_j \cdot W > -v_j(W) - c_j(W), \\ \Longrightarrow \quad & u_j(x'_j) > u_j(x_j). \end{aligned}$$

This contradicts the fact that s is a subgame-perfect equilibrium. □

Lemma 9. *If $W = w_j$, then $x \in \varphi(w,v,c)$.*

Proof. Lemma 8 ensures that $j \in \arg\min_{i \in N}(v_i(W) + c_i(W))$. The choice of j at the beginning of the present section ensures that $\sigma(j) = 1$ and $\sigma(i) = 0$ for each $i \neq j$. Although $p^* = p(w,v,c)$ is not guaranteed in this case where $W = w_j$ and $w_i = 0$ for each $i \neq j$, the outcome function of the game form ensures that $m_i = -p^* \cdot w_i = 0 = -p(w,v,c) \cdot w_i$ for each $i \neq j$ and hence $m_j = -p(w,v,c) \cdot w_j + v_j(W)$. Therefore, x is a fair pricing rule allocation and hence $x \in \varphi(w,v,c)$. □

By Lemmas 7 and 9, for any subgame-perfect equilibrium allocation $x \in SPE(\Gamma(w,c),v)$, we have $x \in \varphi(w,v,c)$. Therefore, the following proposition holds.

Proposition 2. *For all $v \in \mathcal{V}^N$, we have $SPE(\Gamma(w,c),v) \subset \varphi(w,v,c)$.*

Propositions 1 and 2 completes the proof of the theorem.

4. Conclusion

We have proposed a two-stage game form that implements the fair pricing correspondence in subgame-perfect equilibrium. Our game form can be used as a tool for the social planner who wishes to realize the fair pricing rule allocations but does not possess information on disutilities of districts. Our game form is simple in the sense that messages that each district reports are just a price, and 'yes' or 'no'. Furthermore, our game form achieves full implementation. That is, not only every fair pricing rule allocation can be realized as an equilibrium allocation, but also every equilibrium allocation is in fact a fair pricing rule allocation.

It is true that our game form may possess disadvantages. One of them is that the game form depends on information on wastes and construction costs. Although Sakai [2010] points out that this information can be collected, it would be desirable if the game form is defined independently of such information. We are now working to modify our game form and to prove a new result, which will be presented in the future.

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Appendix: Nash implementability of the fair pricing correspondence

We show that the fair pricing correspondence φ is not Nash implementable by presenting an example of a violation of Maskin monotonicity (Maskin [1999]), which is a necessary condition for Nash implementation.

If a NIMBY problem is such that $n = 2$, $w_1 = w_2 = 1$, $v_1(y) = 3y$, $v_2(y) = 2y$, and $c_1(y) = c_2(y) = y$, then $\varphi(w, v, c) = \{x\} \equiv \{(W, \sigma, m)\}$ is such that $W = 2$, $\sigma(1) = 0$, $\sigma(2) = 1$, $m_1 = -3$, and $m_2 = 1$.

Consider another problem for which v_2 is replaced by v'_2 as follows: If a problem is such that $n = 2$, $w_1 = w_2 = 1$, $v_1(y) = 3y$, $v'_2(y) = y$, and $c_1(y) = c_2(y) = y$, then $\varphi(w, v_1, v'_2, c) = \{x'\} \equiv \{(W, \sigma, m')\}$ is such that $W = 2$, $\sigma(1) = 0$, $\sigma(2) = 1$, $m'_1 = -2$, and $m'_2 = 0$.

We note that v'_2 is a Maskin monotonic transformation of v_2 at $x \in \varphi(w, v, c)$ since the following condition holds: For all $(W, \sigma'', m'') \in X(w, c)$,

$$\text{if } -v_2(\sigma''(2) \cdot W) + m''_2 \leq -v_2(\sigma(2) \cdot W) + m_2 = -v_2(2) + 1 = -3$$

$$\text{then } -v_2'(\sigma''(2)\cdot W)+m_2''\leq -v_2'(\sigma(2)\cdot W)+m_2=-v_2'(2)+1=-1.$$

The above condition holds because $\sigma''(2)\cdot W$ is either 0 or 2.[9]

Maskin monotonicity requires that $x\in\varphi(w,v_1,v_2',c)$, but in fact $x\notin\varphi(w,v_1,v_2',c)=\{x'\}$. Therefore, φ violates Maskin monotonicity and hence φ is not Nash implementable.

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[9]When $\sigma''(2)\cdot W=0$, the condition means that if $m_2''\leq -3$ then $m_2''\leq -1$. When $\sigma''(2)\cdot W=2$, the condition means that if $-4+m_2''\leq -3$ then $-2+m_2''\leq -1$.