Ex-ante α-core with communication system in an S-4 logic model

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Abstract

We consider a non-transferable utility game derived from a strategic form game with asymmetric information where each player's information is provided by communication system and define the solution concept of α-core in the game. Our main purpose is to relax the axiom of wisdom and to prove the nestedness of communication system ensure the nonemptiness of the solution concept even if players' information is non-partitional.

Keywords: NTU-game derived from strategic form game, communication system, nestedness, axiom of wisdom, non-partitional information, ex-ante α-core

JEL classification: C79, D82, D85

1 Introduction

This paper is in the stream of the researches on the core of games with asymmetric information. We define a strategic form game with asymmetric information where players' information are given by communication system. We consider that the communication system can give the players non-partitional information structure[1]. We derive the non-transferable utility game (NTU-game) from the strategic form game above and define the solution concept, ex-ante α-core with communication system

[1] Most of the literature assume that each player's information is a partition on the state space

based on the idea of core. Our main purpose is to examine the nonemptiness of the core in relation to the property of communication system.

In the literature, Scarf (1971) defines the core of the NTU-game which is derived from the strategic form game under symmetric information. Scarf (1971) proves that the balancedness of the game is a sufficient condition for the nonemptiness of the core in the general settings. Wilson (1978) defines two types of the core concepts under asymmetric information according to the players' information exchange patterns. The coarse core is based on the idea the players in a coalition can use only the common information among them, while the fine core is based on the idea they can make use of the members' information fully. The nonemptiness of the coarse core is shown in the general settings. Note that Wilson (1978) deals with interim decision making in contrast that this paper considers ex-ante decision making. Yannelis (1991) considers the case where each player uses his or her private information independent of the coalition he or she belongs to. Yannelis (1991) defines the core concept of the game which is called the private core and examines the nonemptiness of the private core.

Maus (2003) introduces the communication system with which we can consider the players' information exchange more flexibly. It exogenously defines each player's information dependent on the coalition he or she belongs to. Maus (2003) derives the NTU-game from an exchange economy and shows that the core concept is nonempty if the communication system is nested[2]. Hirase and Utsumi (2005) extend the seminal work by Maus (2003). They derive the NTU-game from a strategic form game[3] with communication system and show that the core concept is nonempty if the communication system satisfies the nestedness.

All the literature above assume that each player's information is a partition on the state space. In this paper, we would like to relax the axiom of wisdom, which means that players' information can be non-partitional. Our main purpose is to prove the nonemptiness of the core concept of the NTU-game derived from a strategic form game even if the axiom of wisdom is relaxed. This idea comes from Samet (1991) who extends the agreement theorem by Aumann (1976) with relaxing the axiom of wisdom.

The rest of this seminal paper is organized as follows. In Section 2, we define the game, the properties of the communication system, and the core concept. Section 3 discusses the players' information structure. Section 4 provides a sufficient condition for nonemptiness of the core concept. Section 5 gives the concluding remarks and the future problems.

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[2]Nestedness of the communication system implies that each player has the more information in the larger coalition.

[3]A strategic form game is an extention of an exchange economy, since a game can consider the externality.

2 The Game

The basic definition of the game is based on Hirase and Utsumi (2005). The difference is that we allow the case in which the communication system gives the non-partitional information. This section is organized as follows. First, we define the strategic form game with communication system where players can exchange their information. Second, we define the NTU-game derived from the strategic form game with communication system. Third, we define the ex-ante α-core of the NTU-game above.

Now we start with the definition of the strategic form game with communication system.

Definition 1. A *strategic form game with directed link communication system* Γ is a following list of data $(N, \Omega, \{A_i, u_i\}_{i\in N}, \{\mathcal{P}_i^S\}_{i\in N, S\subset N})$.

- $N = \{1, ..., n\}$ is the set of players.
- $\Omega = \{\omega_1, ..., \omega_l\}$ is the finite state space.
- $A_i \subset \Re^{m_i}$ is the set of actions for player i. We assume A_i is a non-empty, convex, and compact set.
- $u_i : \prod_{i\in N}(A_i^\Omega) \to \Re$ is player i's payoff function. We assume u_i is a continuous and quasi-concave function on $\prod_{i\in N}(A_i^\Omega)$.
- $\{\mathcal{P}_i^S\}_{i\in S, S\subset N}$, which we call the *communication system*, determines each player's information dependent on a coalition he or she belongs to. We assume $\mathcal{P}_i^S$ is a function Ω to $2^\Omega \setminus \{\emptyset\}$ for all $S \subset N$ and $i \in S$.

A_i^Ω is the set of functions from Ω to A_i. Σ_i denotes A_i^Ω which we call *the universal strategy set for player i*. $\mathcal{P}_i^{S,g}$ is the information of player i in S when the coalition S is formed.

Definition 2. We define the set of *player i's strategies for information* $\mathcal{P}_i^S$ as follows.

$$\Sigma_i^S := \{\sigma_i^S \in \Sigma_i | \sigma_i^S \text{ is } \mathcal{P}_i^S\text{-measurable}\}.$$

This measurability condition is required because we can not consider each player can take different actions at ω and ω' if he or she do not distinct a state ω from another state ω' with the information $\mathcal{P}_i^{S,g}$.

The set of the joint strategies of the coalition S is described as $\Sigma^S = \prod_{i\in S}\Sigma_i^S$. σ^S denotes the typical element of Σ^S. For all $R \subset N$, a partition on R is interpreted as a *coalition structure of R* .

$P(R)$ denotes the set of all coalition structure of R. That means, for $R \subset N$, $P(R)$ is defined by

$$P(R) := \left\{ \{S_1, ..., S_L\} \;\middle|\; \bigcup_{l=1}^{L} S_l = R \text{ and } S_l \cap S_m = \emptyset, \text{ for all } l, m \in \{1, ..., L\} \text{ s.t. } l \neq m \right\}.$$

Using these notations and these definitions, we can derive an NTU-game from the strategic form game with communication system Γ as follows.

Definition 3. We define *the NTU characteristic function V derived from the strategic form game with communication system* Γ as follows[4].

$$V(S) := \bigcup_{\sigma^S \in \Sigma^S} \bigcap_{Q \in P(N \setminus S)} \bigcap_{\substack{(\sigma^T)_{T \in Q} \\ \in (\Sigma^T)_{T \in Q}}} \left\{ (u_1, ..., u_n) \in \Re^n \;\middle|\; \forall i \in S,\ u_i \leq u_i(\sigma^S, (\sigma^T)_{T \in Q}) \right\}$$

for all $S \in \mathcal{N}$.

$V(S)$ means the set of payoff profiles, which the players in a coalition S can gain at least, even if the worst situation (coalition structure and strategies of $N \setminus S$) for S occurs. This α-concept is suggested by Aumann and Peleg (1960).

Before the definition of the core concept of the NTU-game, we define the improvement concept.

Definition 4. We say that *a coalition S improves upon the payoff vector u in $\Re^n$* if u is in $\mathrm{int}V(S)$.

The definition of the improvement concept implies that all members in the coalition S have the incentive to deviate from the situation where the players gain the payoff profiles u. All members of S can be better off by the deviation. We can not say u is stable in this sense, and u can not be a candidate for a solution concept. We define the solution concept of the NTU-game which is robust for such a deviation.

Definition 5. For an NTU characteristic function V, we define *the ex-ante α-core with communication system $C(V)$* as

$$C(V) := V(N) \setminus \bigcup_{T \subset N} \mathrm{int}V(T).$$

The ex-ante α-core with communication system is the set of the payoff profiles that is achieved by the grand coalition and is not improved upon by any coalition of the players. Without confusion, we

[4] The definition is same as Hirase and Utsumi (2005).

use the word just “core” to refer to the ex-ante α-core with communication system.

3 The Information Structure

In this section, we discuss the players' information structure in detail. In the literature introduced in Section 1, each player's information is assumed to be a partition on Ω and we would like to relax it. For this purpose, we consider 3 properties of the players information as follows. Note that we omit the index of the players and coalitions, $\mathcal{P}_i^S$ is simply denoted by $\mathcal{P}$.

- **P-1 :** $\omega \in \mathcal{P}(\omega)$ for all ω in Ω.
- **P-2** : $\omega' \in \mathcal{P}(\omega)$ implies $\mathcal{P}(\omega') \subset \mathcal{P}(\omega)$ for all ω in Ω.
- **P-3** : $\omega' \in \mathcal{P}(\omega)$ implies $\mathcal{P}(\omega') \supset \mathcal{P}(\omega)$ for all ω in Ω.

P-1 is the property that each player never excludes the real state. When the real state is ω, the player thinks that ω may have occurred.

To clarify the meaning of P-2, we consider its contraposition: If there is a state z such that $z \notin \mathcal{P}(\omega)$ and $z \in \mathcal{P}(\omega')$, then $\omega' \notin \mathcal{P}(\omega)$. This can be interpreted as follows. If a player at ω knows that the state z is impossible and that if the real state is ω', then z is probable, then he or she infers that ω' is not the real state.

To clarify the meaning of P-3, consider its contraposition again: If there is a state z such that $z \in \mathcal{P}(\omega)$ and $z \notin \mathcal{P}(\omega')$, then $\omega' \notin \mathcal{P}(\omega)$. This means that if a player at ω knows that the state z is probable and that if the real state is ω', then z is impossible, then he or she infers that ω' is not the real state. P-3 is called *axiom of wisdom.* We consider the situation where P-3 is not necessarily satisfied.

Note that the following remark holds.

Remark 1. If P-1 and P-3 are satisfied, then P-2 is also satisfied.

Proof. Suppose $\omega' \in \mathcal{P}(\omega)$. Then P-3 implies $\mathcal{P}(\omega') \supset \mathcal{P}(\omega)$. Since $\omega \in \mathcal{P}(\omega)$ by P-1, we have $\omega \in \mathcal{P}(\omega')$. Then exchanging the role of ω and ω' in P-3, we can conclude $\mathcal{P}(\omega') \subset \mathcal{P}(\omega)$. $\square$

We discuss the relation between these properties and the partitional information structure. We say $\mathcal{P}$ is *partitional* if there is a partition on Ω such that for any $\omega \in \Omega$ the set $\mathcal{P}(\omega)$ is equal to the element of the partition that contains ω.

As in Rubinstein (1998), the following proposition holds.

Proposition 1.

$\mathcal{P}$ is partitional if and only if $\mathcal{P}$ satisfies P-1, P-2, and P-3.

Proof. If $\mathcal{P}$ is partitional, $\mathcal{P}$ obviously satisfies P-1, P-2, and P-3.

On the other hand, suppose that $\mathcal{P}$ satisfies P-1, P-2, and P-3. If $\mathcal{P}(\omega)$ and $\mathcal{P}(\omega')$ intersect and $z \in \mathcal{P}(\omega) \cap \mathcal{P}(\omega')$ then by P-2 and P-3, we obtain $\mathcal{P}(\omega) = \mathcal{P}(\omega') = \mathcal{P}(z)$. From P-1, we obtain $\bigcup_{\omega\in\Omega} \mathcal{P}(\omega) = \Omega$. Thus, $\mathcal{P}$ is partitional. □

This proposition implies that most of the literature assume players' information satisfies P-1, P-2, and P-3[5]. However, if the rationality of the players is more bounded, all the property of the information is not necessarily satisfied. Some partitional information structure and non-partitional information structure are displayed in the following examples

Example 1.

Suppose $\mathcal{P}$ is as follows.

$\mathcal{P}(\omega_1) = \mathcal{P}(\omega_2) = \{\omega_1, \omega_2\}$, $\mathcal{P}(\omega_3) = \mathcal{P}(\omega_4) = \{\omega_3, \omega_4\}$, and $\mathcal{P}(\omega_5) = \mathcal{P}(\omega_6) = \{\omega_5, \omega_6\}$.

We can check that P-1, P-2, and P-3 are satisfied, hence this information is partitional and described as in Figure 1.

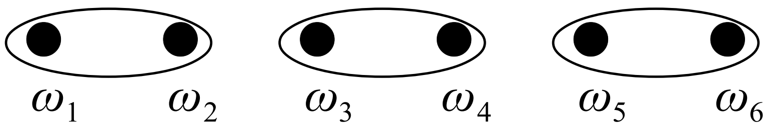

Figure 1: Partitional Information Structure

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Example 2.

Suppose $\mathcal{P}'$ is as follows.

$\mathcal{P}'(\omega_1) = \mathcal{P}'(\omega_2) = \{\omega_1, \omega_2\}$, $\mathcal{P}'(\omega_3) = \mathcal{P}'(\omega_4) = \{\omega_1, \omega_3, \omega_3, \omega_4\}$, and $\mathcal{P}'(\omega_5) = \mathcal{P}'(\omega_6) = \{\omega_5, \omega_6\}$.

We can find that P-1 and P-2 are satisfied and P-3 is not satisfied ($\omega_1 \in \mathcal{P}'(\omega_3)$ but $\mathcal{P}'(\omega_1) \not\supset \mathcal{P}'(\omega_3)$). Hence this information is non-partitional and described as in Figure 2.

[5]Note that, in the literature of modal logic, the model where each players information structure satisfies P-1, P-2, and P-3 is called S-5.

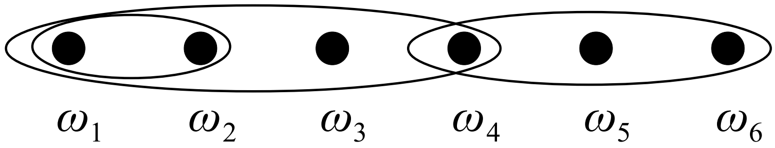

Figure 2: Non-Partitional Information Structure

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Rubinstein (1998) shows the example where the information structure is not partitional.

Example 3. (Rubinstein; 1998)

One gets the good/badl results of an examination. He or she forgets bad news and remembers good news. Denoting good news as G and bad news as B, $\Omega = \{G, B\}$, $\mathcal{P}(G) = \{G\}$, $\mathcal{P}(B) = \{G, B\}$. This information satisfies P-1 and P-2. However, it does not satisfy P-3. $G \in \mathcal{P}(B)$, $B \in \mathcal{P}(B)$ but $B \notin \mathcal{P}(G)$. At B, the player does not conclude that the state is B from the absence of knowledge of the good news.

This situation can be described as in Figure 3.

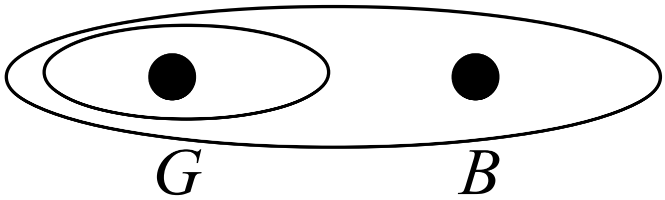

Figure 3: Non-Partitional Information Structure

$\diamondsuit$

Example 4.

Suppose $\mathcal{P}''$ is as follows.

$\mathcal{P}''(\omega_1) = \mathcal{P}''(\omega_2) = \{\omega_1, \omega_2\}$, $\mathcal{P}''(\omega_3) = \mathcal{P}''(\omega_4) = \{\omega_3, \omega_4\}$, and $\mathcal{P}''(\omega_5) = \mathcal{P}''(\omega_6) = \{\omega_4, \omega_5, \omega_6\}$

We can check that P-2 is not satisfied either ($\omega_4 \in \mathcal{P}(\omega_5)$ but $\mathcal{P}(\omega_4) \not\subset \mathcal{P}(\omega_5)$). This information is not partitional and described as in Figure 4.

We discuss relaxing the axioms and nonemptiness of the core in the next section.

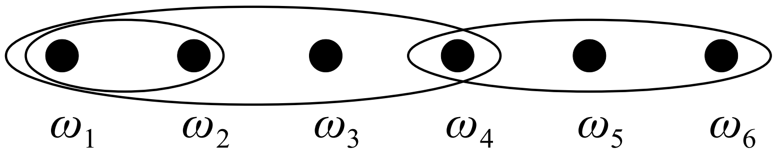

Figure 4: Non-Partitional Information Structure

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4 The Nonemptiness of the Core

We analyze the relation between the property of the communication system and the nonemptiness of the core in this section.

We provide the definition of the nestedness of the communication system.[6]

Definition 6.

A communication system $\{\mathcal{P}_i^S\}_{i\in S, S\subset N}$ is *nested* if for all S and $T \subset \mathcal{N}$ such that $S \subset T$,

$$\mathcal{P}_i^S(\omega) \supset \mathcal{P}_i^T(\omega) \quad \text{for all } i \in S \text{ and } \omega \in \Omega.$$

The nestedness implies that each player in the larger coalition has the richer information. Hirase and Utsumi (2005) show the nestedness of the communication system ensures the nonemptiness of the core when each player's information structure is partitional. Combining this fact and the remark in the previous section, we can obtain the following proposition.

Proposition 2.

The α-core with communication system of V is non-empty if the communication system is nested and satisfies P-1 and P-3.

We can also say the nonemptiness of the core is ensured even if the property P-3 is not satisfied. That means, the following proposition holds. Note that, in the literature of modal logic, the model where each players information structure satisfies P-1 and P-2 is called S-4.

Proposition 3.

The α-core with communication system of V is non-empty if the communication system is nested

[6] The definition of the nestedness is same as Maus (2003)

and satisfies P-1 and P-2.[7]

This means that the core can be nonempty even if players' information is described as in Example 2 and 3. Therefore, we can obtain the nonemptiness of the core even if the players information is not partitional. However note that Example 4 is not the case because its information does not satisfy P-2.

5 Concluding Remarks

We can interpret our main result as the extensions of some seminal works in the literature. Our model is an asymmetric information version of Scarf (1971) and our framework allows non-partitional information structure which is not considered by Hirase and Utsumi (2005).

We can consider the following points as future problems.

At first, we would like to extend the results of Hirase (2009 and 2015). Hirase (2009) examines the partial cooperation situation by using the network[8]. The model deals with the game where players can cooperate and exchange information only through the network which is called network communication system and given exogenously. It is proved that the core concept is nonempty if network communication system is nested. Hirase (2015) focuses on the properties of the links of the network communication system and examines the nonemptiness of the core concept. Both papers assume the partitional information structure and we would like to relax it.

Second, we would like to describe how the directed links are formed, that is, to consider the model in which the structure of the links is endogenously determined. Considering asymmetric information versions of the models by Hirase (2012 and 2013), Jackson and Wolinsky (1996) and Watts (2001) can be the case.

Third, information exchange with our communication system does not necessarily satisfy incentive compatible constraints of the players. We would like to define and examine incentive compatible solution concept. Applying the ideas by Forges and Minelli (2001) and Yazar (2001) to our model can be the case.

At last, we would like to consider more bounded rationality of the players, which means to relax the axiom(s) on players' information more. If it is possible, it is an extension of this seminal work.

[7] The outline of the proof is same as Hirase and Utsumi (2005).

[8] The idea is based on Myerson (1977).

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